JOURNAL OF APPROXIMATION THEORY 57, 245–258 (1989)

Hermite–Padé Polynomials Associated with Certain Generalized Hypergeometric Functions*

J. NUTTALL

Department of Physics, University of Western Ontario, London, Ontario, Canada N6A 3K7

Communicated by Oved Shisha

Received January 31, 1985

1. INTRODUCTION

In this paper we apply the properties of Riemann modules [2] to construct both types of Hermite–Padé polynomials [5] associated with the functions 1, ${}_pF_{p-1}(\substack{a \\ c}; x)$, and $p-1$ additional functions contiguous with this generalized hypergeometric function. We introduce the notion of an augmented module and use it, together with arguments suggested by Chudnovsky [2], to show that the remainder element $R(x)$ for type I polynomials satisfies a generalized hypergeometric differential equation. This leads to an explicit expression for the type I polynomials, and a similar procedure is carried out for the type II case. The results for type II polynomials coincide with a formula for one of the polynomials previously deduced by Chudnovsky [1, 3].

The work might lead to the derivation of the asymptotics for the Hermite–Padé polynomials which might shed some light on the author's conjectures on this subject [5].

2. AUGMENTED MODULES

Suppose we have a Riemann module [2] of dimension p with monodromy matrices $V_1, V_2, \ldots$ corresponding to singular points $b_1, b_2, \ldots$. Then another module, the augmented module, of dimension $m = p + 1$, can be constructed with the same singular points and monodromy matrices $\mathbf{V}_j$,

$$\mathbf{V}_j = \begin{pmatrix} V_j, & (I - V_j)\underline{1} \\ \underline{0}^T, & 1 \end{pmatrix}, \tag{2.1}$$

* Supported in part by the Natural Sciences and Engineering Research Council of Canada.

0021-9045/89 $3.00

where $\underset{\sim}{0}$, $\underset{\sim}{1}$ are column matrices consisting of p 0's and 1's, respectively. If $\underset{\sim}{W}(x)$,

$$\underset{\sim}{W}(x)=\begin{pmatrix} W_1(x) \\ \vdots \\ W_p(x) \end{pmatrix}, \tag{2.2}$$

is an element of the original module, then

$$W(x)=\begin{pmatrix} \underset{\sim}{W}(x) \\ 0 \end{pmatrix} \tag{2.3}$$

is an element of the augmented module. Also in the augmented module there is

$$\begin{pmatrix} \underset{\sim}{1} \\ 1 \end{pmatrix}. \tag{2.4}$$

The eigenvalues of $\mathbf{V}_j$ are the eigenvalues of V_j together with the value unity.

The augmented dual module corresponds to the same singular points and matrices $\tilde{\mathbf{V}}_j$,

$$\tilde{\mathbf{V}}_j=(\mathbf{V}_j^T)^{-1}=\begin{pmatrix} \tilde{V}_j, & \underset{\sim}{0} \\ \underset{\sim}{1}^T(I-\tilde{V}_j), & 1 \end{pmatrix}, \tag{2.5}$$

where $\tilde{V}_j=(V_j^T)^{-1}$ are the matrices for the dual of the original module. If $\underset{\sim}{\tilde{W}}(x)$ is an element of the original dual module then

$$\begin{pmatrix} \underset{\sim}{\tilde{W}}(x) \\ -\underset{\sim}{1}^T\underset{\sim}{\tilde{W}}(x) \end{pmatrix} \tag{2.6}$$

belongs to the augmented module, as does

$$\begin{pmatrix} \underset{\sim}{0} \\ 1 \end{pmatrix}. \tag{2.7}$$

3. Type I H.P. Polynomials Corresponding to a Certain Augmented Module

It has been stated [2], and we have shown explicitly [5], that ${}_pF_{p-1}\left({a \atop c}; x\right)$ and functions contiguous thereto are the first components of elements in a certain module of dimension p with singular points $0, 1, \infty$.

This fact may be used to construct type I H.P. polynomials corresponding to p of the contiguous functions. We now wish to do the same for the augmented module constructed from this module.

We restrict attention to the diagonal case, where $m = p + 1$ polynomials of degree n, $P(j, x)$, are chosen to satisfy

$$\sum_{j=1}^{m} F_1(j, x)\, P(j, x) = O(x^{m(n+1)-1}). \tag{3.1}$$

For $F_1(j, x)$ we take

$$\begin{aligned} F_1(1, x) &= {}_pF_{p-1}\left({a \atop c}; x\right) \\ F_1(j, x) &= \delta^{(j-1)} F_1(1, x), \qquad j = 2, ..., p \\ F_1(m, x) &= 1, \end{aligned} \tag{3.2}$$

where $\delta = x(d/dx)$. These functions are all first components of elements of the augmented module described above. The complete elements are given by

$$\begin{aligned} F(j, x) &= \begin{pmatrix} \underline{F}(j, x) \\ 0 \end{pmatrix}, \qquad j = 1, ..., p \\ F(m, x) &= \begin{pmatrix} \underline{1} \\ 1 \end{pmatrix}, \end{aligned} \tag{3.3}$$

where $\underline{F}(j, x)$ is the appropriate element in the original module.

Explicitly we have [5, Eq. (4.7.18)]

$$\underline{F}(1, x) = KU\underline{Y}^{(\infty)}(x), \tag{3.4}$$

where

$$Y_k^{(\infty)}(x) = x^{-a_k}\, {}_pF_{p-1}\left({1 - c_1 + a_k, \quad . \quad . \quad . \quad , 1 - c_p + a_k \atop 1 - a_1 + a_k, \ ..., [1 - a_k + a_k], ..., 1 - a_p + a_k}; x^{-1} \right),$$
$$k = 1, ..., p. \tag{3.5}$$

We take $c_1 = 1$ and assume that no pair of $\{c_j\}$, $\{a_j\}$ differs by an integer. (In (3.5) the term in [ ] is omitted from the list.) The matrices K, U are given by

$$\begin{aligned} K_{jk} &= \exp(2\pi i(j-1)a_k), \\ U_{jk} &= H_{1j}\,\delta_{jk}, \end{aligned} \qquad j, k = 1, ..., p, \tag{3.6}$$

where H is given in [5, Sect. 4.7.1]. The element $\underline{F}(j, x)$ is obtained by applying $\delta^{(j-1)}$ to $\underline{F}(1, x)$.

We define another element $R(x)$ in the augmented module by

$$R(x) = x^{-m(n+1)+1} \sum_{j=1}^{m} F(j, x)\, P(j, x) \tag{3.7}$$

and use Riemann's arguments, as propounded by Chudnovsky, to find a differential equation satisfied by each component of $R(x)$. The exponents of ${}_pF_{p-1}(x)$ at its singularities are

$$\begin{aligned} &x = 0: 0, 1 - c_2, ..., 1 - c_p \\ &x = \infty: a_1, a_2, ..., a_p \\ &x = 1: 0, 1, ..., p-2, d, \qquad d = \sum_{j=2}^{p} c_j - \sum_{j=1}^{p} a_j = s_c - s_a. \end{aligned} \tag{3.8}$$

From this we can deduce the difference between each of the exponents of $R(x)$ and a corresponding one from the following list is a nonnegative integer:

$$\begin{aligned} &x = 0: -\eta + 1, -\eta + 2 - c_2, ..., -\eta + 2 - c_p, 0 \\ &x = \infty: a_1 + \bar{\eta}, ..., a_p + \bar{\eta}, \bar{\eta} \\ &x = 1: 0, 0, ..., 0, d - p + 1. \end{aligned} \tag{3.9}$$

We have set $\eta = m(n+1)$, $\bar{\eta} = p(n+1)$. The exponents of $\delta^k R$ obey the same relations except that at $x = 0, 1$, the last entries must be replaced by $1, d - p - k + 1$, respectively.

Now let $(-1)^k \Delta_k(x)$ be the determinant of the matrix formed by removing the kth column of the matrix

$$(R, \delta R, \delta^2 R, ..., \delta^{p+1} R). \tag{3.10}$$

A familiar argument [2] shows that, near $x = 0$, $\Delta_1(x)$ has the behavior

$$\Delta_1(x) = x^{-p\eta + 2p - s_c} \times (\text{function analytic near } x = 0). \tag{3.11}$$

For $\Delta_k(x)$, $k = 2, ..., m+1$, the power of x is decreased by 1. Near $x = \infty$, the dominant factor is $x^{-s_a - p\eta}$ for all k, while at $x = 1$, we have

$$\begin{aligned} &(x-1)^{d-2p}, \qquad k = 1, ..., m \\ &(x-1)^{d-2p+1}, \qquad k = m+1. \end{aligned} \tag{3.12}$$

It follows, since $\Delta_k(x)$ is analytic in x except at the three singularities, that

$$\begin{aligned}\Delta_1(x) &= x^{-p\eta+2p-s_c}(x-1)^{d-2p}\lambda_1\\ \Delta_k(x) &= x^{-p\eta+2p-s_c-1}(x-1)^{d-2p}\,\Pi_k(x), \qquad k=2, ..., m \\ \Delta_{m+1}(x) &= x^{-p\eta+2p-s_c-1}(x-1)^{d-2p+1}\lambda_{m+1},\end{aligned} \tag{3.13}$$

where

$$\Pi_k(x) = -\mu_k + \lambda_k x, \qquad k=2, ..., m \tag{3.14}$$

and $\{\lambda_k\}$, $\{\mu_k\}$ are constants.

For now we assume that $\lambda_{m+1} \neq 0$ and normalize so that $\lambda_{m+1}=1$. Later it will be shown that R is unique (up to a constant factor) and that indeed $\lambda_{m+1} \neq 0$.

We deduce that each component of $R(x)$ satisfies

$$(x-1)\,\delta^m R(x) + \sum_{k=1}^{p} \Pi_k(x)\,\delta^k R(x) + x\lambda_1 R(x) = 0. \tag{3.15}$$

This may be rewritten in the form

$$\{xP(\delta) - Q(\delta)\}\, R(x) = 0, \tag{3.16}$$

where

$$\begin{aligned} P(x) &= \sum_{k=0}^{m} \lambda_{k+1} x^k \\ Q(x) &= \sum_{k=1}^{m} \mu_k x^k, \end{aligned} \tag{3.17}$$

$\mu_m = 1$, the standard form of the differential equation for the generalized hypergeometric function ${}_mF_p(\substack{\alpha \\ \gamma}; x)$.

By comparing with the exponents of $R(x)$ at $x = 0, \infty$, we deduce that α, γ can only be

$$\begin{aligned} &\alpha : a_1 + \bar{\eta}, ..., a_p + \bar{\eta}, \bar{\eta} \\ &\gamma : c_2 + \eta - 1, ..., c_p + \eta - 1, \eta, \end{aligned} \tag{3.18}$$

for no other choice would satisfy the inequality on the exponents at $x = 1$.

We know that each component of $R(x)$ is a linear combination, with constant coefficients, of m independent solutions of (3.16), and we now proceed to obtain these relations, analogous to (3.4). In [5, Sect. 4.7.1], such relations were obtained by using the theorems of Smith [6] that relate (3.5) to a set of expansions about $x = 0$ that satisfy the same differential equation. We follow the same approach for $R(x)$, but care must be taken since two of the parameters in α, γ are now integers, so that the conditions for applying the theorems of Smith [6] are not satisfied.

It is easy to check that independent solutions of (3.16) with a specific singular behavior at the origin are still given by ([6, Eq. (3)] or [5, Eq. (4.74)])

$$\Pi_k^{(0)}(x) = x^{1-\gamma_k} {}_mF_p\left(\begin{matrix}1+\alpha_1-\gamma_k, & . \quad . \quad . \quad . & , 1+\alpha_m-\gamma_k \\ 1+\gamma_1-\gamma_k, & ..., [1+\gamma_k-\gamma_k], ..., & 1+\gamma_m-\gamma_k\end{matrix}; x\right),$$
$$k = 1, ..., m \quad (3.19)$$

with the proviso that, for $k = m$, the series representing ${}_mF_p$ must be terminated after the $(n+1)$th term. Solutions with a given singularity at ∞ are given by the same formula as before

$$\Pi_k^{(\infty)}(x) = x^{-\alpha_k} {}_mF_p\left(\begin{matrix}1-\gamma_1+\alpha_k, & . \quad . \quad . \quad . & , 1-\alpha_m+\alpha_k \\ 1-\alpha_1+\alpha_k, & ..., [1-\alpha_k+\alpha_k], ..., & 1-\alpha_m+\alpha_k\end{matrix}; x^{-1}\right),$$
$$k = 1, ..., 1, m. \quad (3.20)$$

It may be seen that now $\Pi_m^{(0)}(x)$ is proportional to $\Pi_m^{(\infty)}(x)$.

There is a matrix $\mathbf{H}_{jk}$, $j, k = 1, ..., m$, such that

$$\Pi_j^{(0)}(x) = \sum_{k=1}^{m} \mathbf{H}_{jk} \Pi_k^{(\infty)}(x), \qquad j = 1, ..., m, \tag{3.21}$$

where all functions are taken to be single-valued in the complex plane cut along the positive real axis from 1 to ∞. We have immediately that $\mathbf{H}_{mk} = 0$, $k = 1, ..., p$, and $\mathbf{H}_{km} = 0$, $k = 2, ..., p$. The remaining elements of $\mathbf{H}$ may be obtained by taking appropriate limits in the formula of Smith [6, Eq. (5)], and those that are needed are given in the Appendix.

We now come to the main result of this section.

THEOREM 3.1. *The polynomials* $P(j, x)$ *and the remainder element* $R(x)$ *satisfying* (3.1), (3.7), *are unique up to a constant factor, and* $R(x)$ *is given by*

$$\begin{aligned} \underline{R}(x) &= KX\Pi^{(\infty)}(x) + R_m(x)\underline{1} \\ R_m(x) &= \mathbf{H}_{1m}\Pi_m^{(\infty)}(x), \end{aligned} \tag{3.22}$$

where

$$X_{jk} = \mathbf{H}_{1j}\delta_{jk}, \qquad j, k = 1, ..., p. \tag{3.23}$$

Proof. 1. We show that $R(x)$ constructed as above belongs to the augmented module. This requires us to show that

$$\mathbf{V}R(x) = R^c(x), \tag{3.24}$$

where $R^c(x)$ is the continuation of $R(x)$ round the singular point corresponding to monodromy matrix $\mathbf{V}$. It is sufficient to treat $x = 0, \infty$. At $x = \infty$, we have

$$R^c(x) = \begin{pmatrix} KXA^2 \underline{\Pi}^{(\infty)}(x) + R_m(x)\underline{1} \\ R_m(x) \end{pmatrix} \tag{3.25}$$

and, from (2.1),

$$\begin{aligned} \mathbf{V}_\infty R(x) &= \begin{pmatrix} V_\infty \underline{R}(x) + R_m(x)\underline{1} - R_m(x) V_\infty \underline{1} \\ R_m(x) \end{pmatrix} \\ &= \begin{pmatrix} V_\infty KX\underline{\Pi}^{(\infty)}(x) + R_m(x)\underline{1} \\ R_m(x) \end{pmatrix}. \end{aligned} \tag{3.26}$$

We have used A^2 as given in [5, Sect. 4.7.1] by

$$(A^2)_{jk} = \exp(2\pi i a_j)\,\delta_{jk}, \qquad j, k = 1, ..., p, \tag{3.27}$$

and we also showed there that

$$V_\infty = KA^2K^{-1} \tag{3.28}$$

from which (3.24) follows for $x = \infty$.

The argument is similar for $x = 0$. The properties of $\mathbf{H}$ imply that, with

$$\begin{aligned} Z_1^{(0)}(x) &= \Pi_1^{(0)}(x) - \mathbf{H}_{1m}\Pi_m^{(\infty)}(x) \\ Z_k^{(0)}(x) &= \underline{\Pi}_k^{(0)}(x), \qquad k = 2, ..., p, \end{aligned} \tag{3.29}$$

then

$$\underline{Z}^{(0)}(x) = N\underline{\Pi}^{(\infty)}(x), \tag{3.30}$$

where

$$N_{jk} = \mathbf{H}_{jk},\ j, k = 1, ..., p. \tag{3.31}$$

For continuation round $x = 0$,

$$(\underline{Z}^{(0)}(x))^c = C^2\underline{Z}^{(0)}(x), \tag{3.32}$$

where

$$(C^2)_{jk} = \exp(2\pi i c_j)\, \delta_{jk}, \qquad j, k = 1, ..., p. \tag{3.33}$$

Now

$$V_0 = W_0 C^2 W_0^{-1}, \tag{3.34}$$

where

$$W_0 = KUH^{-1}. \tag{3.35}$$

Using

$$(\underline{R}(x))^c = KXN^{-1}C^2 N\Pi^{(\infty)}(x) + R_m(x)\underline{1} \tag{3.36}$$

we find that, for $x=0$, (3.24) will hold provided that

$$V_0 KX = KXN^{-1}C^2 N \tag{3.37}$$

and from our formulae it can be checked that (3.37) holds. It follows that $R(x)$ as constructed belongs to the augmented module.

2. Now we show that the functions $P(j, x)$ obtained by solving

$$\sum_{j=1}^{m} F_k(j, x)\, P(j, x) = x^{\eta - 1} R_k(x), \qquad k = 1, ..., m \tag{3.38}$$

are polynomials of degree n that satisfy (3.1). Since $F_m(j, x) = 0$, $j = 1, ..., p$, we immediately obtain

$$P(m, x) = x^{\eta - 1} R_m(x) \tag{3.39}$$

which is a polynomial of degree n. The remaining functions satisfy

$$\sum_{j=1}^{p} \underline{F}(j, x)\, P(j, x) = x^{\eta - 1} KX\Pi^{(\infty)}(x) \tag{3.40}$$

and are given by

$$P(j, x) = \Lambda_j(x)/\Lambda_m(x), \qquad j = 1, ..., p, \tag{3.41}$$

where $(-1)^j \Lambda_j(x)$ is the determinant of the matrix formed by omitting the jth column from

$$[\underline{F}(1, x), \underline{F}(2, x), ..., \underline{F}(p, x), KX\underline{\Pi}^{(\infty)}(x)]. \tag{3.42}$$

Our arguments above show that $KX\Pi^{(\infty)}(x)$ is an element of the original module and the standard procedure may be used to evaluate the determinants. In particular we find

$$\Lambda_m(x) = x^{p-1-s_c}(x-1)^{d-p+1}\ \text{const.}, \tag{3.43}$$

where the constant is non-zero. We conclude that $P(j, x)$ as given by (3.41) are polynomials of degree n. The polynomials satisfy (3.1) since $R_1(x) = \Pi_1^{(0)}(x)$ is analytic near $x = 0$.

3. To prove uniqueness, we first argue that, as seen by evaluation near $x = \infty$, λ_{m+1} calculated with the $R(x)$ given above does not vanish. Suppose that there is an independent set of polynomials corresponding to remainder element $R^*(x)$. The exponents of $R^*(x)$ at $x = \infty$ must be no less than those of $R(x)$. It follows that, for sufficiently small $\mu \neq 0$, λ_{m+1} calculated for $R(x) + \mu R^*(x)$ is not zero. The uniqueness of the form and solution of (3.16), which holds if $\lambda_{m+1} \neq 0$, leads to the desired result.

4. Type II H.P. Polynomials

We can use a similar method of reasoning to reproduce Chudnovsky's results [1, 3] on type II polynomials associated with the same augmented module and also to obtain new formulae. We consider only the diagonal case where we have polynomials $Q(j, x)$, $j = 1, ..., m$,

$$F_1(i, x)\, Q(j, x) - F_1(j, x)\, Q(i, x) = O(x^{mn+1}), \qquad i, j = 1, ..., m. \tag{4.1}$$

Because for each n the diagonal type I polynomials correspond to normal points in the lattice of dimensions, it follows [4] that the polynomials $Q(j, x)$ are unique up to a constant factor.

Define the functions $T_j(x)$, $j = 1, ..., m$, as the solution of

$$Q(j, x) = \sum_{k=1}^{m} F_k(j, x)\, T_k(x), \qquad j = 1, ..., m. \tag{4.2}$$

It follows that, under continuation round a branch point, $T_j(x)$, $j = 1, ..., m$, transform as an element in the augmented dual module. Since $\det(F_k(j, x))$, $j, k = 1, ..., m$, is, from (3.43), analytic and not zero away from branch points, we see that $T_j(x)$, $j = 1, ..., m$, are indeed the components of an element $T(x)$ in the dual module.

We could now proceed to deduce the exponents of $T(x)$ at the singular points and show as before that the components satisfy an equation of the form (3.16). It turns out that, instead of α, γ, the sets of parameters $\tilde{\alpha}, \tilde{\gamma}$ must be used, where

$$\begin{aligned} &\tilde{\alpha}: -a_1 - pn, \ -a_2 - pn, ..., \ -a_p - pn, \ -pn \\ &\tilde{\gamma}: 1 - c_2 - mn, ..., 1 - c_p - mn, \ -mn. \end{aligned} \tag{4.3}$$

It is not necessary to follow this procedure because we already have uniqueness, so we just state the form of $T(x)$ and show that $\{Q(j, x)\}$ constructed from (4.2) are polynomials that satisfy (4.1).

Independent solutions of the generalized hypergeometric differential equation (3.16) corresponding to parameters $\tilde{\alpha}$, $\tilde{\gamma}$, with specific behavior at $0, \infty$ are $\tilde{\Pi}_k^{(0)}(x)$, $\tilde{\Pi}_k^{(\infty)}(x)$, $k = 1, ..., m$, obtained from (3.19), (3.20) by changing $\alpha \to \tilde{\alpha}$, $\gamma \to \tilde{\gamma}$. In this case the series representing $\tilde{\Pi}_1^{(0)}(x)$ must be terminated after the $(pn + 1)$th term, so that this solution is a polynomial of degree pn. It is proportional to $\tilde{\Pi}_m^{(\infty)}(x)$. Again the two sets of solutions are linearly related. Using the method of Smith the relations can be put, using

$$\begin{aligned} &\tilde{Z}_1^{(0)}(x) = \tilde{\Pi}_m^{(0)}(x) + \lambda \tilde{\Pi}_m^{(\infty)}(x) \\ &\tilde{Z}_k^{(0)}(x) = \tilde{\Pi}_k^{(0)}(x), \qquad k = 2, ..., p, \end{aligned} \tag{4.4}$$

in the form

$$\underline{\tilde{Z}}^{(0)}(x) = \tilde{N} \underline{\Pi}^{(\infty)}(x). \tag{4.5}$$

The values of λ, $\tilde{N}_{jk}$, $j, k = 1, ..., p$, are given in the Appendix.

For our results we shall need functions $Y_k^{(0)}(j, x)$, $j, k = 1, ..., p$, given by

$$Y_k^{(0)}(1, x) = x^{1 - c_k} {}_pF_{p-1} \left(\begin{matrix} 1 + a_1 - c_k, & . & . & . & , 1 + a_p - c_k \\ 1 + c_1 - c_k, & ..., & [1 + c_k - c_k], & ..., & 1 - c_p - c_k \end{matrix} ; x \right),$$
$$k = 1, ..., p, \tag{4.6}$$

and

$$\underline{Y}^{(0)}(j, x) = \delta^{(j-1)} \underline{Y}^{(0)}(1, x). \tag{4.7}$$

An alternative way of writing (3.4) gives

$$\underline{F}(j, x) = W_0 \underline{Y}^{(0)}(j, x). \tag{4.8}$$

In [5, Sect. 4.7.1] we used the notation $\bar{U}$, $\bar{H}$, etc., to indicate that the signs of $a_1, ..., a_p, c_2, ..., c_p$ had been changed. We also defined the diagonal $p \times p$ matrix D.

THEOREM 4.1. *The polynomial* $Q(m, x)$ *is given by*

$$Q(m, x) = d_1 \lambda \tilde{\Pi}_m^{(\infty)}(x) \tag{4.9}$$

and $T(x)$ *by*

$$\begin{aligned} \underline{T}(x) &= (K^T)^{-1} D\bar{U}\bar{H}^{-1} D_0 \tilde{\underline{Z}}^{(0)}(x) \\ T_m(x) &= Q(m, x) - \underline{1}^T \underline{T}(x), \end{aligned} \tag{4.10}$$

where D_0 *is a diagonal* $p \times p$ *matrix. An expression for* $Q(j, x)$, $j = 1, ..., p$, *alternative to* (4.2), *is*

$$Q(j, x) = \sum_{k=1}^{p} Y_k^{(0)}(j, x) \tilde{Z}_k^{(0)}(x) d_k. \tag{4.11}$$

The form of $\{d_k\}$ *is given in* (4.24).

Proof. 1. We first show that $T(x)$ is an element of the augmented dual module which, from (2.6), (2.7), follows if $\underline{T}(x)$ belongs to the original dual module. In [5] we proved that

$$\tilde{V}_0 = \tilde{W}_0 \bar{C}^2 \tilde{W}_0^{-1}, \tag{4.12}$$

where

$$\tilde{W}_0 = (K^T)^{-1} D\bar{U}\bar{H}^{-1}. \tag{4.13}$$

For any diagonal D_0 we have, for continuation round $x = 0$,

$$\begin{aligned} \tilde{V}_0 \underline{T}(x) &= \tilde{V}_0 \tilde{W}_0 D_0 \tilde{\underline{Z}}^{(0)}(x) \\ &= \tilde{W}_0 \bar{C}^2 D_0 \tilde{\underline{Z}}^{(0)}(x) \\ &= \underline{T}^c(x) \end{aligned} \tag{4.14}$$

as required.

We also found

$$\tilde{V}_\infty = \tilde{W}_\infty \bar{A}^2 \tilde{W}_\infty^{-1} \tag{4.15}$$

with

$$\tilde{W}_\infty = (K^T)^{-1} D. \tag{4.16}$$

The relation

$$\tilde{V}_\infty \underline{T}(x) = \underline{T}^c(x) \tag{4.17}$$

for continuation round $x=\infty$ implies

$$\bar{A}^2\bar{U}\bar{H}^{-1}D_0\tilde{N}=\bar{U}\bar{H}^{-1}D_0\tilde{N}\bar{A}^2. \tag{4.18}$$

Since $\bar{A}^2$ is diagonal with no two diagonal elements the same, this implies that

$$\bar{U}\bar{H}^{-1}D_0\tilde{N}=\text{diagonal}=\bar{U}D_1 \tag{4.19}$$

say, i.e.,

$$D_0\tilde{N}=\bar{H}D_1. \tag{4.20}$$

In the Appendix we give diagonal matrices D_0, D_1 satisfying (4.20), which proves that $\underline{T}(x)$ is in the dual module.

2. Now (4.2) may be written as

$$Q(j,x)=\underline{F}^T(j,x)\,\underline{T}(x), \qquad j=1,...,p, \tag{4.21}$$

which must, on account of the transformation properties, be rational with poles possible only at $x=0, 1, \infty$. A study of the exponents of $\underline{F}(j,x)$, $\underline{T}(x)$ shows that (4.21) is a polynomial of degree pn as required.

3. To prove (4.1) we first derive (4.11). In [5, Appendix] we showed that $W_0^T\tilde{W}_0$ was a diagonal matrix called Q which we rename D_2 here. We showed that

$$(D_2)_{jj}=M_{jj}\bar{B}_j\bar{M}_{jj}, \tag{4.22}$$

where M_{jj}, B_j are given in [5, Eqs. (A1.13), (A1.17)]. From (4.8) and (4.10) we have

$$\begin{aligned}Q(j,x)&=\underline{F}^T(j,x)\,\underline{T}(x)\\&=(\underline{Y}^{(0)}(x))^T\,W_0^T\tilde{W}_0D_0\underline{\tilde{Z}}^{(0)}(x), \qquad j=1,...,p\end{aligned} \tag{4.23}$$

which is of the form (4.11) with

$$d_k=(D_0)_{kk}\,(D_2)_{kk}. \tag{4.24}$$

Now

$$F_1(j,x)=Y_1^{(0)}(j,x), \qquad j=1,...,p, \tag{4.25}$$

so that

$$F_1(i,x)\,Q(j,x)-F_1(j,x)\,Q(i,x)$$
$$=\sum_{k=2}^{p}\left(Y_1^{(0)}(i,x)\,Y_k^{(0)}(j,x)-Y_1^{(0)}(j,x)\,Y_k^{(0)}(i,x)\right)\tilde{Z}_k^{(0)}(x)\,d_k,$$
$$i,j=1,\ldots,p$$
$$=O(x^{mn+1}) \tag{4.26}$$

from (4.4), (4.6), and (4.11).

Also

$$F_1(m,x)\,Q(j,x)-F_1(j,x)\,Q(m,x)$$
$$=\sum_{k=2}^{p}Y_k^{(0)}(j,x)\,\tilde{Z}_k^{(0)}(x)\,d_k$$
$$+\,Y_1^{(0)}(j,x)[d_1(\tilde{\Pi}_m^{(0)}(x)+\lambda\tilde{\Pi}_m^{(\infty)}(x))-d_1\lambda\tilde{\Pi}_m^{(\infty)}(x)],\qquad j=1,\ldots,p$$
$$=O(x^{mn+1}). \tag{4.27}$$

This proves the theorem.

Appendix

By following the method of Smith's proof [6], we find

$$\mathbf{H}_{jk}=e^{i\pi(\alpha_k-\gamma_j+1)}\left[\frac{\Gamma(\gamma_j-\alpha_k)\,\Gamma(1+\gamma_k-\gamma_j)}{\Gamma(\gamma_k-\alpha_k)}\right]\left[\prod_{\substack{t=1\\t\neq k}}^{m}\frac{\Gamma(\alpha_t-\alpha_k)\,\Gamma(1+\gamma_t-\gamma_j)}{\Gamma(\gamma_t-\alpha_k)\,\Gamma(1+\alpha_t-\gamma_j)}\right],$$
$$j,k=1,\ldots,p \tag{A1}$$

$$\mathbf{H}_{1m}=e^{i\pi\alpha_m}\frac{[\prod_{t=1}^{p}\Gamma(\alpha_t-\alpha_m)][\prod_{t=1}^{m}\Gamma(\gamma_t]}{[\prod_{t=2}^{m}\Gamma(\gamma_t-\alpha_m)][\prod_{t=1}^{p}\Gamma(\alpha_t)]}. \tag{A2}$$

Similarly it is found that

$$\lambda=(-1)^n\left[\prod_{t=1}^{p}\frac{\Gamma(\tilde{\alpha}_t-\tilde{\alpha}_m)\,\Gamma(1+\tilde{\gamma}_t-\tilde{\gamma}_m)}{\Gamma(\tilde{\gamma}_t-\tilde{\alpha}_m)\,\Gamma(1+\tilde{\alpha}_t-\tilde{\gamma}_m)}\right] \tag{A3}$$

and

$$\tilde{N}_{1k}=e^{i\pi(\tilde{\alpha}_k-\tilde{\gamma}_m+1)}\left[\frac{\Gamma(\tilde{\gamma}_m-\tilde{\alpha}_k)\,\Gamma(1+\tilde{\gamma}_k-\tilde{\gamma}_m)}{\Gamma(\tilde{\gamma}_k-\tilde{\alpha}_k)}\right]$$
$$\times\left[\prod_{\substack{t=1\\t\neq k}}^{m}\frac{\Gamma(\tilde{\alpha}_t-\tilde{\alpha}_k)\,\Gamma(1+\tilde{\gamma}_t-\tilde{\gamma}_m)}{\Gamma(\tilde{\gamma}_t-\tilde{\alpha}_k)\,\Gamma(1+\tilde{\alpha}_t-\tilde{\gamma}_m)}\right],\qquad k=1,\ldots,p \tag{A4}$$

$$\tilde{N}_{jk} = e^{i\pi(\tilde{\alpha}_k - \tilde{\gamma}_j + 1)} \left[\frac{\Gamma(\tilde{\gamma}_j - \tilde{\alpha}_k)\, \Gamma(1 + \tilde{\gamma}_k - \tilde{\gamma}_j)}{\Gamma(\tilde{\gamma}_k - \tilde{\alpha}_k)} \right]$$

$$\times \left[\prod_{\substack{t=1 \\ t \neq k}}^{m} \frac{\Gamma(\tilde{\alpha}_t - \tilde{\alpha}_k)\, \Gamma(1 + \tilde{\gamma}_t - \tilde{\gamma}_j)}{\Gamma(\tilde{\gamma}_t - \tilde{\alpha}_k)\, \Gamma(1 + \tilde{\alpha}_t - \tilde{\gamma}_j)} \right], \qquad j = 2, ..., p;\, k = 1, ..., p. \tag{A5}$$

By considering $\tilde{N}_{jk}/\bar{H}_{jk'}$ with $\bar{H}_{jk}$ obtained from [5, Eqs. (4.7.7), (4.7.9)], we find that (4.20) holds with

$$(D_0)_{jj} = \left[\prod_{t=1}^{m} \frac{\Gamma(1 + \tilde{\alpha}_t - \tilde{\alpha}_j)}{\Gamma(1 + \tilde{\gamma}_t - \tilde{\gamma}_j)} \right] \left[\prod_{t=1}^{p} \frac{\Gamma(1 + \bar{c}_t - \bar{c}_j)}{\Gamma(1 + \bar{a}_t - \bar{c}_j)} \right], \qquad j = 2, ..., p \tag{A6}$$

$$(D_1)_{jj} = \frac{\Gamma(\bar{c}_j - \bar{a}_j)}{\Gamma(\tilde{\gamma}_j - \tilde{\alpha}_j)} \left[\prod_{\substack{t=1 \\ t \neq j}}^{m} \frac{\Gamma(\tilde{\alpha}_t - \tilde{\alpha}_j)}{\Gamma(\tilde{\gamma}_t - \tilde{\alpha}_j)} \right] \left[\prod_{\substack{t=1 \\ t \neq j}}^{p} \frac{\Gamma(\bar{c}_t - \bar{a}_j)}{\Gamma(\bar{a}_t - \bar{a}_j)} \right], \qquad j = 2, ..., p. \tag{A7}$$

To obtain $(D_0)_{11}$, $(D_1)_{11}$, replace $\tilde{\alpha}_j$, $\tilde{\gamma}_j$ by $\tilde{\alpha}_m$, $\tilde{\gamma}_m$.

REFERENCES

1. G. V. CHUDNOVSKY, Padé approximations to the generalized hypergeometric functions, I, *J. Math. Pures Appl.* **58** (1979), 445–476.
2. G. V. CHUDNOVSKY, Padé approximation and the Riemann monodromy problem, *in* "Bifurcation Phenomena in Mathematical Physics and Related Topics" (C. Bardos and D. Bessis, Eds.), pp. 449–510, Reidel, Dordrecht, 1980.
3. G. V. CHUDNOVSKY, Padé approximations to generalized hypergeometric functions, II, unpublished.
4. J. H. LOXTON AND A. J. VAN DER POORTEN, Multi-dimensional generalizations of the Padé table, *Rocky Mountain J. Math.* **9** (1979), 385–393.
5. J. NUTTALL, Asymptotics of diagonal Hermite–Padé polynomials, *J. Approx. Theory* **42** (1984), 299–386.
6. F. C. SMITH, Relations among the fundamental solutions of the generalized hypergeometric equation when $p = q + 1$. I. Non-logarithmic cases, *Bull. Amer. Math. Soc.* **44** (1938), 429–433.